Quadratic Programs for High Relative Degree Spatial Constraints and Spatiotemporal Specifications with Spacecraft Applications

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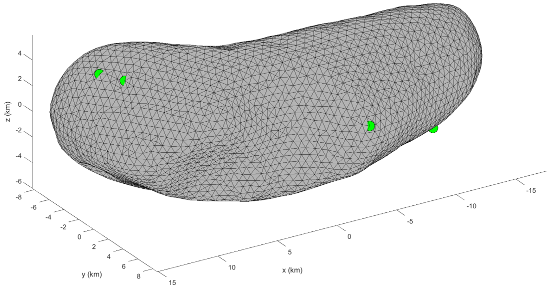
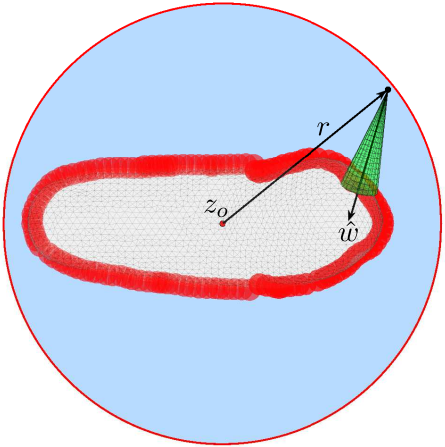
- Safety-critical spacecraft autonomy





- Control Barrier Functions (CBFs) have recently been used across disciplines for safety-critical control
- CBFs combined with other objectives and Quadratic Programs (QPs) to generate control inputs online

$$u = \arg\min_{Au \leq b} u^{\mathrm{T}} H(x) u + F(x) u$$

Background - Relative Degree

- For dynamics
$$\dot{x} = f(x) + g(x)u$$
and safe set
$$S = \{x \in \mathbb{R}^n \mid h(x) \leq 0\},$$
a typical CBF constraint is
$$\dot{h}(x) = L_f h(x) + L_g h(x)u \leq \alpha(-h(x))$$
- This does not work if h is of high relative degree, i.e. $L_g h(x)u \equiv 0, \forall u \in \mathbb{R}^m$

Prior Work

- Methods for converting high relative degree h to CBFs
  - Backstepping approach (Hsu, Xu, Ames, ACC 2015)
  - Exponentials CBFs (Nguyen, Sreenath, ACC 2016)
  - Higher Order CBFs (Xiao, Belta, CDC 2019)
- CBF applications to objectives with time specifications
  - Time Varying CBFs (Lindemann, Dimarogonas, CSS Letters 2019)
  - Prescribed Time CLFs (Garg, Arabi, Panagou, CDC 2019)
  - CBFs in Planning (Yang, Belta, Tron, ACC 2020)

Motivation and Contribution

- Goal is to develop an online control method that
  1. Guarantees safe set invariance for high relative degree h_s
  2. Accomplishes prescribed time goals for high relative degree h_g
  3. Avoids converting h_s, h_g into CBFs and choosing a class-$\mathcal{K}$ function

Problem Formulation

- Dynamics $\dot{x} = f(x) + g(x)u, x \in \mathbb{R}^n, u \in \mathbb{R}^m$
- Safe set $S_s = \{x \in \mathbb{R}^n \mid h_s^{[i]}(x) \leq 0, \forall i = \{1, \cdots, p\}\}$
- Goal sets $S_g^{[j]} = \{x \in \mathbb{R}^n \mid h_g^{[j]}(x) \leq 0\}$ and time intervals $[t_1^{[j]}, t_2^{[j]}]$, $j \in \{1, \cdots, q\}$
- Problem: Determine $u(t)$ online such that $x(t) \in S_s, \forall t \geq t_0$ and $x(t) \in S_g^{[j]}$ over the intervals $[t_1^{[j]}, t_2^{[j]}], \forall j$ for high relative degree h_s, h_g.

Notation

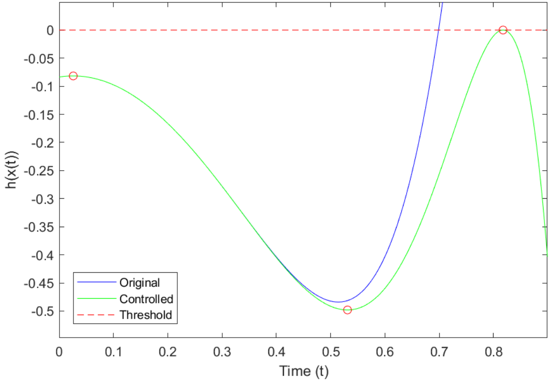
- $h_s =$ safety set defining function
- $h_g =$ goal set defining function
- $c^{[i]} = i^{\text{th}}$ element of c
- $c^{(i)} = i^{\text{th}}$ time derivative of c

- If a subscript omitted, then the equation applies to both h_s and h_g
- If an index is omitted, then any index may be used

- Our strategy is to assign a constant rate a_q to the first controllable derivative $h^{(r)}$ (i.e. $L_g L_f^{r-1} h(x) u \neq 0$)

$$a_q(h, x) = \underset{a \in \mathbb{R}_{\leq 0}}{\arg\min} |a|$$

$$\text{s.t.} \quad \max_{t \geq 0} \left(\frac{1}{r!} a t^r + \sum_{i=1}^{r-1} \frac{1}{i!} h^{(i)}(x) t^i \right) \leq 0$$



Strategy

Example

Suppose h is of relative degree 2. That is,

$$\ddot{h}(x) = L_f^2 h(x) + L_g L_f h(x) u, \quad \exists C \subseteq \mathbb{R}^n : L_g L_f h(x) u \neq 0, \forall x \in C$$

Then a_q satisfies

$$\max_t \left(\frac{1}{2} a_q t^2 + \dot{h}(x) t + h(x) \right) = 0$$

$$\implies a_q = \frac{\dot{h}(x)^2}{2h(x)}$$

Strategy

- The constraint $\ddot{h}(x,u) = a_q(h,x)$ or $\ddot{h}(x,u) \le a_q(h,x)$ is affine in u
  - i.e. $\ddot{h}(x,u) \le a_q(h,x)$ is of form $Au \le b$:

$$L_g L_f h(x) u \le a_q(h,x) - L_f^2 h(x)$$

    so u must have a component parallel to $L_g L_f h(x)^{\mathrm{T}}$
  - Components of u orthogonal to $L_g L_f h(x)^{\mathrm{T}}$ are unconstrained

- Define set $S_{s,\epsilon}^{[i]} = \{x \in X \mid 0 \geq h_s^{[i]}(x) \geq -\epsilon^{[i]}\}$ for some $\epsilon^{[i]} > 0$.

Figure: Visualization of a constraint function's sublevel set and ϵ-boundary set $S_{s,\epsilon}^{[i]}$

Main Results - One Constraint

Assumption 1

The initial state $x_0 = x(t_0)$ is such that $x_0 \in S_s$ and if $x_0 \in \mathrm{bd}(S_s)$, then $f(x_0) \in T(x_0, S_s)$.

- One can ensure satisfaction of a single constraint as follows:

Theorem 1

Let $M < \infty$. If u is chosen such that

$$\ddot{h}_s^{[i]}(x,u) \leq \begin{cases} 0 & h_s^{[i]}(x) = 0 \\ M & h_s^{[i]}(x) \neq 0,\ \dot{h}_s^{[i]}(x) < 0 \\ a_q(h_s^{[i]}, x) & h_s^{[i]}(x) \neq 0,\ \dot{h}_s^{[i]}(x) \geq 0 \end{cases}$$

$\forall x \in S_{s,\epsilon}^{[i]}$, *then* $h_s^{[i]}(x(t)) \leq 0,\ \forall t \geq t_0$.

Main Results - Multiple Constraints

- Define the *Boundary Layer* as the set $S_{s,bd,\epsilon} \triangleq S_s \cap \left(\bigcup_{i=1}^{p} S_{s,\epsilon}^{[i]} \right)$

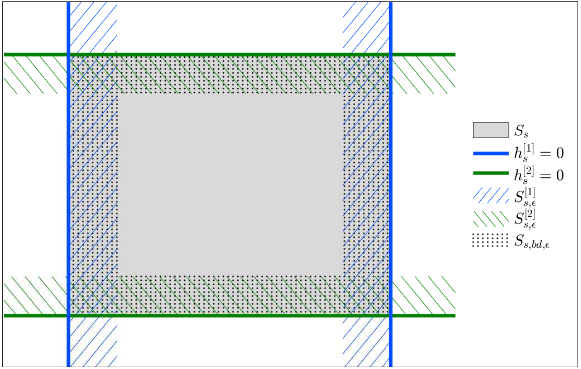


Figure: Visualization of two constraint functions

Proposition 1

Let $S_{s,bd,\epsilon}$ for some $\epsilon \in \mathbb{R}^p_{>0}$ be any Boundary Layer of S_s. If u is chosen such that each $\ddot{h}_s^{[i]}(x,u)$ satisfies Theorem 1, $\forall x \in S_{s,bd,\epsilon}$, $\forall i \in \{1, \cdots, p\}$, then S_s is forward invariant.

- Suppose $x(t_*) \in S_g^{[j]}$. Then staying in $S_g^{[j]}$ is almost the same problem as staying in S_s.
- The difference is that $x(t)$ may leave $S_g^{[j]}$ after $t_2^{[j]}$.

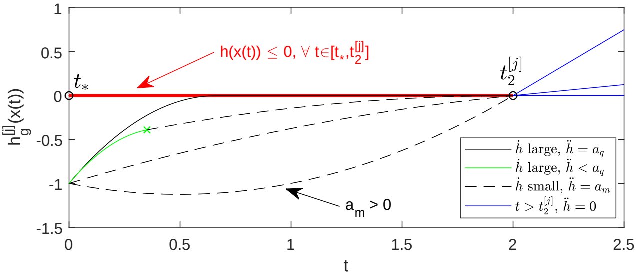


Figure: Possible trajectories of $h_g^{[j]}(x(t))$ when u is chosen as in Theorem 3, both with and without equality, for $t \geq t_*$ for various $\dot{h}_g^{[j]}(x(t_*))$ where $t_* = 0$.

Main Results - Goal Maintenance

- To formalize the notions in the previous figure, define

$$a_m(h, x, t_i, t_f) \triangleq \frac{-2[h(x) + \dot{h}(x)(t_f - t_i)]}{(t_f - t_i)^2}.$$

Assumption 2

The state $x(t_*)$ is such that if $x_* \in \mathrm{bd}(S_g^{[j]})$, then $f(x_*) \in T(x_*, S_g^{[j]})$.

Main Results - Goal Maintenance

Theorem 3

If $h_g^{[j]}(x(t_*)) \leq 0$, $x(t_*)$ *satisfies Assumption 2, and* u *is chosen such that*

$$\ddot{h}_g^{[j]}(x(t),u) \leq \begin{cases} a_q(h_g^{[j]}, x(t)) & \text{if } t - \frac{2h_g^{[j]}(x(t))}{\dot{h}_g^{[j]}(x(t))} \leq t_2^{[j]} \text{ and } \dot{h}_g^{[j]}(x(t)) > 0 \\ a_m(h_g^{[j]}, x(t), t, t_2^{[j]}) & \text{else} \end{cases}$$

$\forall t \in [t_*, t_2^{[j]}]$, *then* $h_g^{[j]}(x(t)) \leq 0, \ \forall t \in [t_*, t_2^{[j]}]$.

- The curve generated by $\ddot{h}_g^{[j]}(x(t), u) \le a_m(h_g^{[j]}, x(t), t, T)$ guarantees that $h_g^{[j]}(x(T)) \le 0$, which can be used to enforce convergence to a goal set.

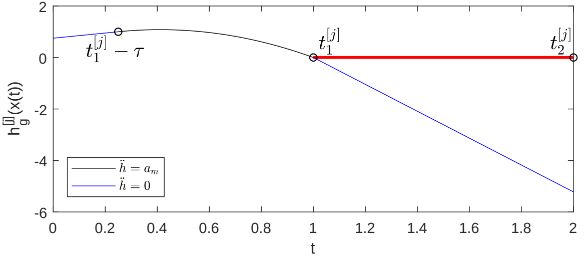


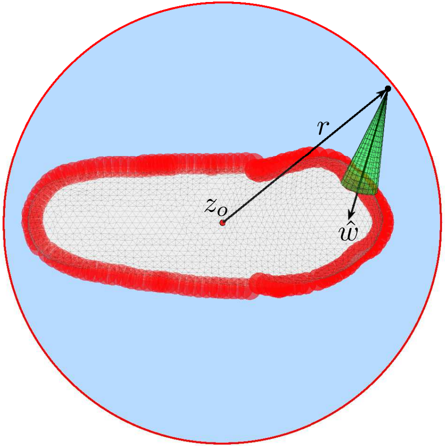
Figure: A plot of $h_g^{[j]}(x(t))$ for a trajectory converging to a goal set

Spacecraft Application

- Spacecraft translation and attitude dynamics

$$\dot{x} = \begin{bmatrix} \dot{r} \\ \dot{v} \\ \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} v \\ f_\mu(r) + u_1 \\ f_{\dot{\theta}}(\theta, \omega) \\ u_2 \end{bmatrix}$$

- Constraints. Let $y = r - z$ for some fixed z
  - Proximity to target: $h_p(x, z) = ||y||^2 - \rho^2 \leq 0$
  - Point at target: $h_b(x, z) = \cos\beta - \hat{w}^{\mathrm{T}}\hat{y} \leq 0$ where $\hat{w}$ is some spacecraft-fixed vector
  - Avoidance of obstacle: $h_a(x, z) = \bar{\rho}^2 - ||y||^2 \leq 0$



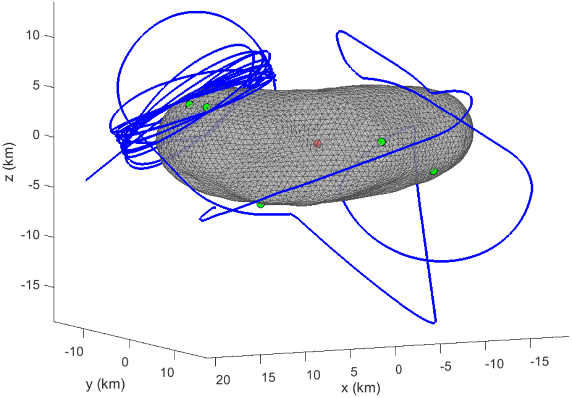
- Quadratic program:

$$u(t) = \underset{\substack{u_1, u_2 \in \mathbb{R}^3 \times \mathbb{R}^3 \\ \delta_p, \delta_b \in \mathbb{R}^q \times \mathbb{R}^q}}{\arg\min} \left(\begin{bmatrix} u_1^{\mathrm{T}} & u_2^{\mathrm{T}} \end{bmatrix} H(x) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + F(x) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \sum_{j=1}^{q} \left[J_{\delta,p} \left(\delta_p^{[j]} \right)^2 + J_{\delta,b} \left(\delta_b^{[j]} \right)^2 \right] \right)$$

subject to

  - Safety constraints
  - Goal convergence constraints
  - Goal maintenace constraints

Spacecraft Results



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Spacecraft Results

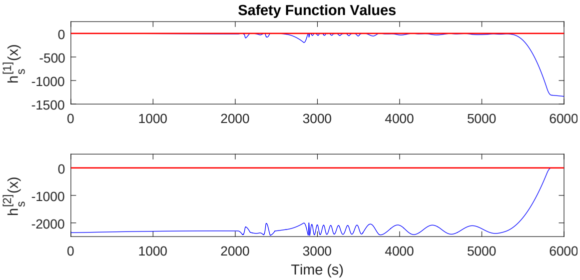

Figure: The values of the safety constraint functions over simulation time.

Spacecraft Results

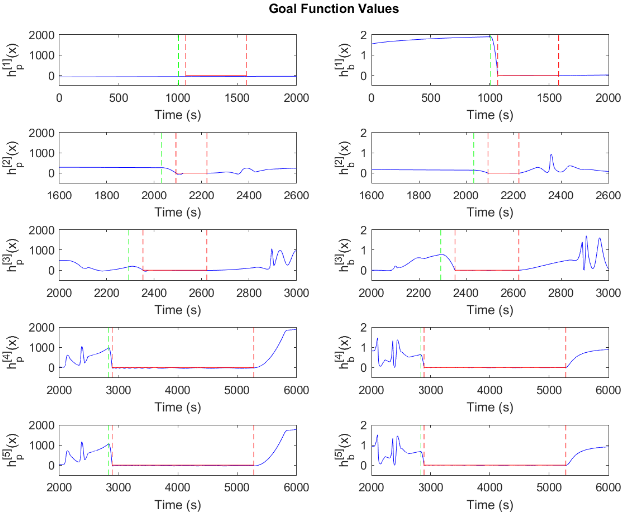

Figure: The values of the goal constraint functions for the 5 targets over simulation time.

Conclusion

- Presented online control method for ensuring safe set invariance and convergence to goal sets in prescribed time
  - Introduced boundary layer
  - Reduced control effort to maintain goal sets
- Current/future work
  - Feasibility under control input constraints/other actuators
  - Fuel-optimality
  - Applications to central gravity

Acknowledgements

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