Modeling Spacecraft Attitude with Quaternions

Joseph Breeden

July 26, 2021

1 Geometry with Quaternions

1.1 Quaternion Properties

It is assumed that the reader is already familiar with the distinction between a physical matrix and a resolved matrix as in [1]. In brief, a *physical matrix*, such as a *physical rotation matrix* $\vec{R}$, is an operator that modifies a *physical vector* $\vec{v}$. A resolved matrix, such as a *rotation matrix* $\mathcal{R}$ or *orientation matrix* $\mathcal{O}$ is a 2D array of numbers. In particular, orientation matrices are a tool for transforming the coordinates of a *math vector* $\bar{v}$, where a math vector is a 1D column array of numbers. The rest of this text is consistent with the notation of [1].

A *physical quaternion*, denoted $\widetilde{q}$ is a mathematical object consisting of a scalar q_0 and physical vector $\vec{q}$:

$$\widetilde{q} = q_0 + \vec{q} \tag{1}$$

Many texts use bold font $\mathbf{q}$ for quaternions, but that is avoided here to facilitate handwritten calculations. It is common to also present quaternions as columns

$$\widetilde{q} = \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix} . \tag{2}$$

In this text, the scalar element appears first in the column, but this varies from author to author.

A physical quaternion can be resolved in a frame to produce a 4-element column array by resolving the vector portion in the chosen frame

$$\widetilde{q}|_{\mathrm{A}} = \begin{bmatrix} q_0 \\ \vec{q}\Big|_{\mathrm{A}} \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} . \tag{3}$$

Every quaternion $\widetilde{q}$ also has a conjugate quaternion $\widetilde{q}^*$ defined as

$$\widetilde{q}^* = q_0 - \vec{q} . \tag{4}$$

At this point, it is worth noting that quaternions are often described as a "hyper-complex number", $\widetilde{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are each complex numbers. While this is accurate, we find this characterization overly limiting as we may wish to resolve physical quaternions in several frames, similar to physical rotation matrices. To accomplish the same behavior as provided by the hyper-complex number formulation, we introduce the quaternion product operator $\otimes$, which behaves as follows

$$a_0 \otimes b_0 = a_0 b_0 , \tag{5}$$

$$a_0 \otimes \vec{b} = a_0 \vec{b} , \tag{6}$$

$$\vec{a} \otimes \vec{b} = \vec{a} \times \vec{b} - \vec{a} \cdot \vec{b} . \tag{7}$$

It follows that the $\otimes$ operator applied to a quaternion satisfies

$$\widetilde{a} \otimes \widetilde{b} = \left(a_0 + \vec{a}\right) \otimes \left(b_0 + \vec{b}\right) = a_0 b_0 + a_0 \vec{b} + b_0 \vec{a} + \vec{a} \times \vec{b} - \vec{a} \cdot \vec{b} \tag{8}$$

$$\widetilde{a} \otimes \vec{b} = \left(a_0 + \vec{a}\right) \otimes \vec{b} = a_0 \vec{b} + \vec{a} \times \vec{b} - \vec{a} \cdot \vec{b} \tag{9}$$

$$\widetilde{a} \otimes b_0 = \left(a_0 + \vec{a}\right) \otimes b_0 = a_0 b_0 + b_0 \vec{a} \tag{10}$$

The $\otimes$ operator is bilinear, associative, and distributive, but NOT commutative. We can equivalently define the $\otimes$ operator for two resolved quaternions

$$(\widetilde{q} \otimes \widetilde{r})|_{\mathrm{A}} = \widetilde{q}|_{\mathrm{A}} \otimes \widetilde{r}|_{\mathrm{A}} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \otimes \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} q_0 r_0 - q_1 r_1 - q_2 r_2 - q_3 r_3 \\ q_0 r_1 + q_1 r_0 + q_2 r_3 - q_3 r_2 \\ q_0 r_2 - q_1 r_3 + q_2 r_0 + q_3 r_1 \\ q_0 r_3 + q_1 r_2 - q_2 r_1 + q_3 r_0 \end{bmatrix} . \tag{11}$$

The quaternion norm is then defined as

$$\|\widetilde{q}\|^2 = \widetilde{q} \otimes \widetilde{q}^* = \widetilde{q}^* \otimes \widetilde{q} \, . \tag{12}$$

The above equation is one of the few cases where commutativity of $\otimes$ does hold. Our application for quaternions in this text is as a representation of frame rotations, which we will show implies that every quaternion will have norm $\|\widetilde{q}\| = 1$. In other applications, one can normalize a quaternion by multiplying by $\frac{1}{\|\widetilde{q}\|}$. Finally, the quaternion inverse is

$$\widetilde{q}^{-1} = \frac{\widetilde{q}^*}{\|\widetilde{q}\|} \, . \tag{13}$$

The conjugate and the inverse satisfy

$$(\widetilde{q} \otimes \widetilde{r})^* = \widetilde{r}^* \otimes \widetilde{q}^* \, , \tag{14}$$

$$(\widetilde{q} \otimes \widetilde{r})^{-1} = \widetilde{r}^{-1} \otimes \widetilde{q}^{-1} \, . \tag{15}$$

1.2 Frame Rotations

Any two frames $\mathrm{F_A} = \begin{bmatrix} \hat{i}_{\mathrm{A}} & \hat{j}_{\mathrm{A}} & \hat{k}_{\mathrm{A}} \end{bmatrix}$ and $\mathrm{F_B} = \begin{bmatrix} \hat{i}_{\mathrm{B}} & \hat{j}_{\mathrm{B}} & \hat{k}_{\mathrm{B}} \end{bmatrix}$ can be related by a unique physical rotation matrix $\vec{R}_{\mathrm{B/A}}$, such that $\mathrm{F_B} = \vec{R}_{\mathrm{B/A}} \mathrm{F_A}$. Moreover, any rotation can be expressed (non-uniquely) by an axis of rotation $\hat{n}$ and an angle θ. Recall the physical rotation matrix

$$\vec{R}_{\hat{n}}(\theta) = \cos\theta \vec{I} + (1 - \cos\theta)\hat{n}\hat{n}^{\mathrm{T}} + \sin\theta \hat{n}^{\times} \, . \tag{16}$$

An alternative way of storing information about an axis-angle rotation is as a quaternion. Define the axis-angle physical quaternion as

$$\widetilde{q}_{\hat{n}}(\theta) = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\hat{n} \, . \tag{17}$$

It immediately follows that $\|\widetilde{q}_{\hat{n}}(\theta)\| = 1$ for any θ and $\hat{n}$. Since the axis-angle representation of a rotation is not unique, it follows that the quaternion representing a frame rotation is not unique either, unlike rotation matrices. Specifically, $\widetilde{q}$ and $-\widetilde{q}$ both represent the same rotation. For this reason, many applications restrict to quaternions with positive scalar elements, which represent short-way rotations, though none of the following math depends on this choice. The identity quaternion is $\widetilde{q}_I = 1$, which represents no rotation.

Let $\widetilde{q} = \widetilde{q}_{\hat{n}}(\theta)$ be an axis-angle physical quaternion and $\vec{r}$ an arbitrary physical vector, and consider the following expression

$$\widetilde{q} \otimes \vec{r} \otimes \widetilde{q}^* = \left(-\vec{q} \cdot \vec{r} + q_0 \vec{r} + \vec{q} \times \vec{r}\right) \otimes \left(q_0 - \vec{q}\right) \tag{18}$$

$$= -q_0 \vec{q} \cdot \vec{r} + q_0 \vec{r} \cdot \vec{q} + (\vec{q} \times \vec{r}) \cdot \vec{q} + (\vec{q} \cdot \vec{r})\vec{q} + q_0^2 \vec{r} + q_0(\vec{q} \times \vec{r}) - q_0(\vec{r} \times \vec{q}) - (\vec{q} \times \vec{r}) \times \vec{q} \tag{19}$$

$$= (\vec{q}\,\vec{q}^{\mathrm{T}})\vec{r} + q_0^2 \vec{r} + 2q_0(\vec{q} \times \vec{r}) + \vec{q} \times (\vec{q} \times \vec{r}) \tag{20}$$

$$= \left((\vec{q}\,\vec{q}^{\mathrm{T}}) + (1 - \vec{q}^{\mathrm{T}}\vec{q})\vec{I} + 2q_0 \vec{q}^{\times} + \vec{q}^{\times^2} \right) \vec{r} \tag{21}$$

$$= \left(\vec{q}\,\vec{q}^{\mathrm{T}} + (1 - \vec{q}^{\mathrm{T}}\vec{q})\vec{I} + 2\cos\left(\frac{\theta}{2}\right) \vec{q}^{\times} + (\vec{q}\,\vec{q}^{\mathrm{T}} - \left(\vec{q}^{\mathrm{T}}\vec{q}\right)\vec{I}) \right) \vec{r} \tag{22}$$

$$= \left(2\vec{q}\,\vec{q}^{\mathrm{T}} + (1 - 2\vec{q}^{\mathrm{T}}\vec{q})\vec{I} + 2\cos\left(\frac{\theta}{2}\right) \vec{q}^{\times} \right) \vec{r} \tag{23}$$

$$= \left(2\hat{n}\hat{n}^{\mathrm{T}} \sin^2\left(\frac{\theta}{2}\right) + \left(1 - 2\sin^2\left(\frac{\theta}{2}\right)\right)\vec{I} + 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\hat{n}^{\times} \right) \vec{r} \tag{24}$$

$$= \left((1 - \cos\theta)\hat{n}\hat{n}^{\mathrm{T}} + \cos\theta \vec{I} + \sin\theta \hat{n}^{\times} \right) \vec{r} \tag{25}$$

$$= \vec{R}_{\hat{n}}(\theta)\vec{r} \tag{26}$$

That is, a physical quaternion and its conjugate rotate a physical vector identically to a physical rotation matrix. Let $\vec{R}_{\mathrm{B/A}}$ be the rotation from frame $\mathrm{F_A}$ to $\mathrm{F_B}$, and let $\hat{n}$ and θ be such that $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}}(\theta)$. We then use the notation $\widetilde{q}_{\mathrm{B/A}}$ to represent the same rotation $\widetilde{q}_{\mathrm{B/A}} = \widetilde{q}_{\hat{n}}(\theta)$. Suppose there are three frames $\mathrm{F_A}, \mathrm{F_B}, \mathrm{F_C}$. Then it holds that

$$\widetilde{q}_{\mathrm{C/A}} = \widetilde{q}_{\mathrm{C/B}} \otimes \widetilde{q}_{\mathrm{B/A}}\,, \tag{27}$$

which can be proven via the equivalence between $\widetilde{q}_{\mathrm{B/A}}$ and $\vec{R}_{\mathrm{B/A}}$. Recall that $\widetilde{q}_{\mathrm{B/A}}$ and $-\widetilde{q}_{\mathrm{B/A}}$ represent the same rotation, and thus both are equivalent to $\vec{R}_{\mathrm{B/A}}$.

It follows from the axis-angle representation that $\vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \mathcal{R}_{\mathrm{B/A}}$. It similarly follows that $\widetilde{q}_{\mathrm{B/A}}\big|_{\mathrm{B}} = \widetilde{q}_{\mathrm{B/A}}\big|_{\mathrm{A}}$, and thus we define

$$\bar{q}_{\mathrm{B/A}} = \widetilde{q}_{\mathrm{B/A}}\big|_{\mathrm{B}} = \widetilde{q}_{\mathrm{B/A}}\big|_{\mathrm{A}}\,. \tag{28}$$

To convert a resolved quaternion to a resolved rotation matrix, one can use the function

$$\mathcal{R}(\bar{q}) = \begin{bmatrix} 1 - 2q_2^2 - 2q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_0q_2 + 2q_1q_3 \\ 2q_0q_3 + 2q_1q_2 & 1 - 2q_1^2 - 2q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_0q_1 + 2q_2q_3 & 1 - 2q_1^2 - 2q_2^2 \end{bmatrix}, \tag{29}$$

where $\mathcal{R}(\bar{q}_{\mathrm{B/A}}) \equiv \mathcal{R}_{\mathrm{B/A}}$. Note that one may come across equivalent forms of the above matrix in other literature which use the fact that $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

Now consider resolving a quaternion rotation in a frame $\mathrm{F_A}$ as follows.

$$\left(\widetilde{q}_{\mathrm{B/A}} \otimes \vec{r} \otimes \widetilde{q}^{*}_{\mathrm{B/A}}\right)\Big|_{\mathrm{A}} = \widetilde{q}_{\mathrm{B/A}}\big|_{\mathrm{A}} \otimes \vec{r}\Big|_{\mathrm{A}} \otimes \widetilde{q}^{*}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \mathcal{R}_{\mathrm{B/A}}\,\vec{r}\Big|_{\mathrm{A}} \tag{30}$$

Next, note that if $\widetilde{q}_{\mathrm{B/A}} = \widetilde{q}_{\hat{n}}(\theta)$, then $\widetilde{q}^{*}_{\mathrm{B/A}} = \widetilde{q}_{\hat{n}}(-\theta)$, so $\widetilde{q}_{\mathrm{B/A}} = \widetilde{q}^{*}_{\mathrm{A/B}}$. Thus,

$$\bar{q}^{*}_{\mathrm{B/A}} \otimes \vec{r}\Big|_{\mathrm{A}} \otimes \bar{q}_{\mathrm{B/A}} = \widetilde{q}^{*}_{\mathrm{B/A}}\Big|_{\mathrm{A}} \otimes \vec{r}\Big|_{\mathrm{A}} \otimes \widetilde{q}_{\mathrm{B/A}}\big|_{\mathrm{A}} = \widetilde{q}_{\mathrm{A/B}}\big|_{\mathrm{A}} \otimes \vec{r}\Big|_{\mathrm{A}} \otimes \widetilde{q}^{*}_{\mathrm{A/B}}\Big|_{\mathrm{A}} = \mathcal{R}_{\mathrm{A/B}}\,\vec{r}\Big|_{\mathrm{A}} = \mathcal{O}_{\mathrm{B/A}}\,\vec{r}\Big|_{\mathrm{A}}\,. \tag{31}$$

Thus, quaternion conjugates provide a means to change the coordinates of a resolved vector from one frame to another frame equivalently to orientation matrices.

Recall that

$$\mathcal{R}_{\mathrm{C/A}} = \vec{R}_{\mathrm{C/A}}\Big|_{\mathrm{C}} = \vec{R}_{\mathrm{C/B}}\Big|_{\mathrm{C}}\,\vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{C}} = \mathcal{R}_{\mathrm{C/B}}\left(\mathcal{O}_{\mathrm{C/B}}\,\vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{B}}\,\mathcal{O}_{\mathrm{C/B}}^{\mathrm{T}}\right) = \mathcal{R}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{C/B}}^{\mathrm{T}} = \mathcal{R}_{\mathrm{B/A}}\mathcal{R}_{\mathrm{C/B}} \tag{32}$$

whereas

$$\mathcal{O}_{\text{C/A}} = \mathcal{R}_{\text{C/A}}{}^{\text{T}} = \left(\mathcal{R}_{\text{B/A}}\mathcal{R}_{\text{C/B}}\right)^{\text{T}} = \mathcal{R}_{\text{C/B}}{}^{\text{T}}\mathcal{R}_{\text{B/A}}{}^{\text{T}} = \mathcal{O}_{\text{C/B}}\mathcal{O}_{\text{B/A}} \tag{33}$$

It similarly holds that

$$\overline{q}_{\text{C/A}} = \widetilde{q}_{\text{C/A}}\big|_{\text{C}} = \widetilde{q}_{\text{C/B}}\big|_{\text{C}} \otimes \widetilde{q}_{\text{B/A}}\big|_{\text{C}} = \overline{q}_{\text{C/B}} \otimes \left(\overline{q}^{*}_{\text{C/B}} \otimes \widetilde{q}_{\text{B/A}}\big|_{\text{B}} \otimes \overline{q}_{\text{C/B}}\right) = \overline{q}_{\text{B/A}} \otimes \overline{q}_{\text{C/B}} \tag{34}$$

and

$$\overline{q}^{*}_{\text{C/A}} = \left(\overline{q}_{\text{B/A}} \otimes \overline{q}_{\text{C/B}}\right)^{*} = \overline{q}^{*}_{\text{C/B}} \otimes \overline{q}^{*}_{\text{B/A}} \tag{35}$$

That is, both orientation matrices and resolved quaternion conjugates follow the "slash-and-split" rule, while resolved rotation matrices and resolved quaternions do not follow this rule.

Note that all of the above rotation formulas assumed that $\widetilde{q}$ was an axis-angle rotation and therefore $\|\widetilde{q}\| = 1$. If instead $\|\widetilde{q}\| \neq 1$, then $\widetilde{q}^{*}$ should be replaced with $\widetilde{q}^{-1}$. In practice, if $\overline{q}$ is the result of numerical integration, then over time $\|\overline{q}\|$ could diverge from 1, in which case using $\overline{q}^{-1}$ in place of $\overline{q}^{*}$ may be appropriate. However, it is often a better choice to enforce that $\|\overline{q}\| = 1$ within the integration method.

2 Kinematics with Quaternions

It is difficult to directly differentiate the scalar and vector components of a quaternion, so we will instead use the quaternion rotation formula as a starting point. Let $\widetilde{q} = \widetilde{q}_{\text{B/A}}$ for two frames F_{A} and F_{B}, where we omit the subscript for compactness. Let $\vec{r}$ be a vector fixed in frame F_{A}. Define another vector

$$\vec{s} = \vec{R}(\widetilde{q})\,\vec{r} = \widetilde{q} \otimes \vec{r} \otimes \widetilde{q}^{*}\,. \tag{36}$$

It follows that

$$\vec{r} = \widetilde{q}^{*} \otimes \vec{s} \otimes \widetilde{q}\,. \tag{37}$$

The derive of $\vec{s}$ is as follows [2].

$$\overset{\text{A}\bullet}{\vec{s}} = \overset{\text{A}\bullet}{\widetilde{q}} \otimes \vec{r} \otimes \widetilde{q}^{*} + \widetilde{q} \otimes \overset{\text{A}\bullet}{\not{\vec{r}}} \otimes \widetilde{q}^{*} + \widetilde{q} \otimes \vec{r} \otimes \overset{\text{A}\bullet}{\widetilde{q}^{*}} \tag{38}$$

$$\overset{\text{B}\bullet}{\not{\vec{s}}} + \vec{\omega}_{\text{B/A}} \times \vec{s} = \overset{\text{A}\bullet}{\widetilde{q}} \otimes \left(\widetilde{q}^{*} \otimes \vec{s} \otimes \widetilde{q}\right) \otimes \widetilde{q}^{*} + \widetilde{q} \otimes \left(\widetilde{q}^{*} \otimes \vec{s} \otimes \widetilde{q}\right) \otimes \overset{\text{A}\bullet}{\widetilde{q}^{*}} \tag{39}$$

$$\vec{\omega}_{\text{B/A}} \times \vec{s} = \overset{\text{A}\bullet}{\widetilde{q}} \otimes \widetilde{q}^{*} \otimes \vec{s} + \vec{s} \otimes \widetilde{q} \otimes \overset{\text{A}\bullet}{\widetilde{q}^{*}} \tag{40}$$

$$= \left(\dot{q}_0 + \overset{\text{A}\bullet}{\vec{q}}\right) \otimes \left(q_0 - \vec{q}\right) \otimes \vec{s} + \vec{s} \otimes \left(q_0 + \vec{q}\right) \otimes \left(\dot{q}_0 - \overset{\text{A}\bullet}{\vec{q}}\right) \tag{41}$$

$$= \Bigg(\underbrace{\left(\dot{q}_0 q_0 + \overset{\text{A}\bullet}{\vec{q}} \cdot \vec{q}\right)}_{=\frac{1}{2}(\widetilde{q} \otimes \overset{\text{A}\bullet}{\widetilde{q}^{*}})=0} - \dot{q}_0 \vec{q} + q_0 \overset{\text{A}\bullet}{\vec{q}} - \overset{\text{A}\bullet}{\vec{q}} \times \vec{q}\Bigg) \otimes \vec{s}$$

$$+ \vec{s} \otimes \Bigg(\underbrace{\left(\dot{q}_0 q_0 + \overset{\text{A}\bullet}{\vec{q}} \cdot \vec{q}\right)}_{=\frac{1}{2}(\widetilde{q} \otimes \overset{\text{A}\bullet}{\widetilde{q}^{*}})=0} + \dot{q}_0 \vec{q} - q_0 \overset{\text{A}\bullet}{\vec{q}} + \vec{q} \times \overset{\text{A}\bullet}{\vec{q}}\Bigg) \tag{42}$$

$$= \left(-\dot{q}_0 \vec{q} + q_0 \overset{\text{A}\bullet}{\vec{q}} - \overset{\text{A}\bullet}{\vec{q}} \times \vec{q}\right) \otimes \vec{s} + \vec{s} \otimes \left(\dot{q}_0 \vec{q} - q_0 \overset{\text{A}\bullet}{\vec{q}} + \vec{q} \times \overset{\text{A}\bullet}{\vec{q}}\right) \tag{43}$$

$$= (-\dot{q}_0 \vec{q} + q_0 \overset{\text{A}\bullet}{\vec{q}} - \overset{\text{A}\bullet}{\vec{q}} \times \vec{q}) \times \vec{s} + \vec{s} \times (\dot{q}_0 \vec{q} - q_0 \overset{\text{A}\bullet}{\vec{q}} + \vec{q} \times \overset{\text{A}\bullet}{\vec{q}}) \tag{44}$$

$$= 2(-\dot{q}_0 \vec{q} + q_0 \overset{\text{A}\bullet}{\vec{q}} - \overset{\text{A}\bullet}{\vec{q}} \times \vec{q}) \times \vec{s} \tag{45}$$

$$= 2(\overset{A\bullet}{\widetilde{q}} \otimes \widetilde{q}^*) \times \vec{s} \tag{46}$$

Since this holds for all $\vec{r}$ and all $\widetilde{q}$, we conclude

$$\vec{\omega}_{B/A} = 2 \overset{A\bullet}{\widetilde{q}} \otimes \widetilde{q}^* \tag{47}$$

$$\frac{1}{2}\vec{\omega}_{B/A} \otimes \widetilde{q} = \overset{A\bullet}{\widetilde{q}} \tag{48}$$

We can also take the derivative as observed in frame F_B as follows.

$$\overset{B\bullet}{\widetilde{q}} = \overset{A\bullet}{\widetilde{q}} - \vec{\omega}_{B/A} \times \vec{q} \tag{49}$$

$$= \frac{1}{2}\omega_{B/A} \otimes \widetilde{q} - \vec{\omega}_{B/A} \times \vec{q} \tag{50}$$

$$= -\frac{1}{2}\vec{q} \cdot \vec{\omega} + \frac{1}{2}q_0\vec{\omega} + \frac{1}{2}\vec{\omega} \times \vec{q} - \omega \times \vec{q} \tag{51}$$

$$= -\frac{1}{2}\vec{q} \cdot \vec{\omega} + \frac{1}{2}q_0\vec{\omega} - \frac{1}{2}\vec{\omega} \times \vec{q} \tag{52}$$

$$= \frac{1}{2}\widetilde{q} \otimes \vec{\omega}_{B/A} \tag{53}$$

The orientation $\bar{q}_{B/A}$ of a body fixed frame F_B with respect to some other frame F_A component is frequently a state variable in a dynamical system, so we require an expression for its derivative. Recall that

$$\overset{A\bullet}{\widetilde{q}}_{B/A}\Big|_A = \dot{\left(\widetilde{q}_{B/A}\big|_A\right)} = \dot{\bar{q}}_{B/A} = \dot{\left(\widetilde{q}_{B/A}\big|_B\right)} = \overset{B\bullet}{\widetilde{q}}_{B/A}\Big|_B . \tag{54}$$

Thus, we have two equivalent formulas for $\dot{\bar{q}}_{B/A}$. Most often, the angular velocity $\vec{\omega}_{B/A}$ is measured in the body fixed frame F_B, so it is more convenient to derive $\dot{\bar{q}}_{B/A}$ from $\overset{B\bullet}{\widetilde{q}}_{B/A}$, though both of the above formulas appear in other literature.

Let $\bar{\omega}_{B/A}(t) = \vec{\omega}_{B/A}\Big|_B = \begin{bmatrix}\omega_1(t)\\ \omega_2(t)\\ \omega_3(t)\end{bmatrix}$. Then the quaternion kinematic equation becomes

$$\dot{\bar{q}}_{B/A} = \frac{1}{2}\underbrace{\begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}}_{=W(t)} \bar{q}_{B/A} . \tag{55}$$

Note that $W(t)$ is skew-symmetric, which is one of the rare cases when time-varying linear equations can be exactly integrated [3]:

$$\bar{q}_{B/A}(t_1) = \exp\left(\frac{1}{2}\int_{t_0}^{t_1} W(t)\right)\bar{q}_{B/A}(t_0) . \tag{56}$$

The above equation is useful because it maintains $\|\bar{q}_{B/A}(t_1)\| = \|\bar{q}_{B/A}(t_0)\|$. That said, for nonsymmetric bodies, Euler's equations of rotational motion (a variant of which is presented in the following section) do not yield closed form expressions for $\bar{\omega}_{B/A}(t)$, so we usually still need to employ numerical integration methods.

3 Spacecraft Attitude Dynamics

We now consider the dynamics for the spacecraft system. Suppose the spacecraft consists of a single rigid body (easily generalized to multiple rigid bodies) containing r reaction wheels, and subject to external torque $\vec{L}_z$ (thrusters, solar radiation pressure, air drag, etc.). Let $J_{\mathcal{B}/z}$ be the moment of inertia of the spacecraft without reaction wheels with respect to some point z (either a point with zero-inertial-acceleration or the center of mass of the combined body+wheels system). Suppose each wheel has axis $\hat{a}_i$ through center of mass c_i, and suppose the moment of inertia of the wheels satisfies

$$\vec{J}_{\mathcal{W}_i/c_i} = \vec{J}^{\perp}_{\mathcal{W}_i/c_i} + J^{\parallel}_{\mathcal{W}_i/c_i}\hat{a}_i\hat{a}_i^{\mathrm{T}} \tag{57}$$

where $J^{\parallel}_{\mathcal{W}_i/c_i}$ is a scalar, and $\vec{J}^{\perp}_{\mathcal{W}_i/c_i}$ is a matrix satisfying $\vec{J}^{\perp}_{\mathcal{W}_i/c_i}\hat{a}_i = \vec{0}$ [4]. Then define

$$\vec{J}_{\mathcal{W}_i/z} = \underbrace{\vec{J}^{\perp}_{\mathcal{W}_i/c_i} - m_i \vec{r}^{\times^2}_{z/c_i}}_{=\vec{J}^{\perp}_{\mathcal{W}_i/z}} + J^{\parallel}_{\mathcal{W}_i/c_i}\hat{a}_i\hat{a}_i^{\mathrm{T}} \tag{58}$$

Let $\mathcal{C}$ denote the complete system composed of bodies $\mathcal{B}$ and $\mathcal{W}_i, i = 1, \cdots, r$. Let $\mathrm{F_B}$ be a frame fixed to $\mathcal{B}$ and $\mathrm{F_A}$ some other frame. Let $\mathrm{F_{W_i}}$ be wheel-fixed frames where $\vec{\omega}_{\mathrm{W}_i/\mathrm{B}} = \Omega_i\hat{a}_i$ for some scalar rate Ω_i. The total angular momentum of the system with respect to $\mathrm{F_A}$ is then

$$\begin{aligned}
\vec{H}_{\mathcal{C}/z/A} &= \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B/A}} + \sum_{i=1}^{r}\vec{J}_{\mathcal{W}_i/z}\vec{\omega}_{\mathrm{W}_i/\mathrm{A}} && (59)\\
&= \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B/A}} + \sum_{i=1}^{r}\left(\vec{J}^{\perp}_{\mathcal{W}_i/z} + J^{\parallel}_{\mathcal{W}_i/c_i}\hat{a}_i\hat{a}_i^{\mathrm{T}}\right)\left(\vec{\omega}_{\mathrm{W}_i/\mathrm{B}} + \vec{\omega}_{\mathrm{B/A}}\right) && (60)\\
&= \underbrace{\left(\vec{J}_{\mathcal{B}/z} + \sum_{i=1}^{r}\vec{J}^{\perp}_{\mathcal{W}_i/z}\right)}_{=\vec{J}_{\mathcal{P}/z}}\vec{\omega}_{\mathrm{B/A}} + \sum_{i=1}^{r}J^{\parallel}_{\mathcal{W}_i/c_i}\hat{a}_i\hat{a}_i^{\mathrm{T}}\left(\Omega_{\mathrm{W}_i/\mathrm{B}}\hat{a}_i + \vec{\omega}_{\mathrm{B/A}}\right) && (61)
\end{aligned}$$

Assuming z is either a ZIA point or the center of mass of $\mathcal{C}$ (which is assumed to be fixed in frame $\mathrm{F_B}$) and $\mathrm{F_A}$ is an inertial frame, it follows that

$$\vec{L}_z = \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{C}/z/\mathrm{A}} = \overset{\mathrm{B}\bullet}{\vec{H}}_{\mathcal{C}/z/\mathrm{A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{H}_{\mathcal{C}/z/\mathrm{A}} \tag{62}$$

We then compute the derivative in frame $\mathrm{F_B}$ as

$$\begin{aligned}
\overset{\mathrm{B}\bullet}{\vec{H}}_{\mathcal{C}/z/\mathrm{A}} &= \vec{J}_{\mathcal{P}/z}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \sum_{i=1}^{r}J^{\parallel}_{\mathcal{W}_i/c_i}\hat{a}_i\hat{a}_i^{\mathrm{T}}\left(\dot{\Omega}_{\mathrm{W}_i/\mathrm{B}}\hat{a}_i + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\right) && (63)\\
&= \underbrace{\left(\vec{J}_{\mathcal{P}/z} + \sum_{i=1}^{r}J^{\parallel}_{\mathcal{W}_i/c_i}\hat{a}_i\hat{a}_i^{\mathrm{T}}\right)}_{=\vec{J}_{\mathcal{C}/z}}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \sum_{i=1}^{r}J^{\parallel}_{\mathcal{W}_i/c_i}\hat{a}_i\dot{\Omega}_{\mathrm{W}_i/\mathrm{B}} && (64)
\end{aligned}$$

Note that while a gyro measures angular velocity of the body with respect to an inertial frame such as $\mathrm{F_A}$, a tachometer such as we usually have on the reaction wheels gives measurements with respect to the mounting frame, in this case the spacecraft bus frame $\mathrm{F_B}$. Thus, it makes sense to use $\Omega_{\mathrm{W}_i/\mathrm{B}}$ as a state instead of adding the rotation $\vec{\omega}_{\mathrm{B/A}}$ of the spacecraft as well. In summary, we have

$$\vec{L}_z = \vec{J}_{\mathcal{C}/z}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{H}_{\mathcal{C}/z/\mathrm{A}} + \sum_{i=1}^{r}J^{\parallel}_{\mathcal{W}_i/c_i}\hat{a}_i\dot{\Omega}_{\mathrm{W}_i/\mathrm{B}} \tag{65}$$

Now, we treat the wheels as their own individual systems. Each reaction wheel is mounted in a housing which permits frictionless rotation about the mounting axis $\hat{a}_i$, and rigidly rotates the wheels with the spacecraft bus about the other two axes. That is, if all the wheels start at rest and we were to manually spin the spacecraft about one of the wheel axes using some other actuator, then the wheel on that axis would stay at rest because there is no friction. Torque on the axis of rotation $\hat{a}_i$ comes purely from the housing electronics and is given by $\vec{L}^{\parallel}_{\mathcal{W}_i} = u_i \hat{a}_i$. Torque in other directions may come from the spacecraft structure and is given by $\vec{L}^{\perp}_{\mathcal{W}_i}$ where $\vec{L}^{\perp}_{\mathcal{W}_i} \cdot \hat{a}_i = 0$. The momentum of each wheel about its axis is

$$\vec{H}^{\parallel}_{\mathcal{W}_i/c_i/\mathrm{A}} = J^{\parallel}_{\mathcal{W}_i/c_i} \hat{a}_i \hat{a}_i^{\mathrm{T}} \left(\Omega_{\mathrm{W}_i/\mathrm{B}} \hat{a}_i + \vec{\omega}_{\mathrm{B/A}} \right) \tag{66}$$

The change in momentum is then

$$\vec{L}_{\mathcal{W}_i} = \vec{L}^{\parallel}_{\mathcal{W}_i} + \vec{L}^{\perp}_{\mathcal{W}_i} = \overset{\mathrm{A}\bullet}{\vec{H}^{\parallel}}_{\mathcal{W}_i/c_i/\mathrm{A}} = \overset{\mathrm{B}\bullet}{\vec{H}^{\parallel}}_{\mathcal{W}_i/c_i/\mathrm{A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{H}^{\parallel}_{\mathcal{W}_i/c_i/\mathrm{A}} \tag{67}$$

$$\vec{L}^{\parallel}_{\mathcal{W}_i} = \overset{\mathrm{B}\bullet}{\vec{H}^{\parallel}}_{\mathcal{W}_i/c_i/\mathrm{A}} \tag{68}$$

$$\vec{L}^{\perp}_{\mathcal{W}_i} = \vec{\omega}_{\mathrm{B/A}} \times \vec{H}^{\parallel}_{\mathcal{W}_i/c_i/\mathrm{A}} \tag{69}$$

We are able to separate the above torques because $\vec{H}^{\parallel}_{\mathcal{W}_i/z/\mathrm{A}}$ is always along $\hat{a}_i$. Thus, $\vec{\omega}_{\mathrm{B/A}} \times \vec{H}^{\parallel}_{\mathcal{W}_i/z/\mathrm{A}}$ is always orthogonal to $\hat{a}_i$. The torque $\vec{L}^{\perp}_{\mathcal{W}_i}$ is not under our control and is therefore not needed for the dynamics derivation, though it may be needed elsewhere to determine the required stiffness of the wheel mountings. Equating the axial torques then yields

$$u_i \hat{a}_i = \vec{L}^{\parallel}_{\mathcal{W}_i} = \overset{\mathrm{B}\bullet}{\vec{H}^{\parallel}}_{\mathcal{W}_i/c_i/\mathrm{A}} = J^{\parallel}_{\mathcal{W}_i/c_i} \hat{a}_i \hat{a}_i^{\mathrm{T}} \left(\dot{\Omega}_{\mathrm{W}_i/\mathrm{B}} \hat{a}_i + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \right) \tag{70}$$

$$u_i = J^{\parallel}_{\mathcal{W}_i/c_i} \left(\dot{\Omega}_{\mathrm{W}_i/\mathrm{B}} + \hat{a}_i \cdot \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \right) \tag{71}$$

In summary (see [5]), all combined we have

$$\vec{J}_{\mathcal{C}/z} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \sum_{i=1}^{r} J^{\parallel}_{\mathcal{W}_i/c_i} \hat{a}_i \dot{\Omega}_{\mathrm{W}_i/\mathrm{B}} = \vec{L}_z - \vec{\omega}_{\mathrm{B/A}} \times \vec{H}_{\mathcal{C}/z/\mathrm{A}} \tag{72}$$

$$J^{\parallel}_{\mathcal{W}_i/c_i} \hat{a}_i \cdot \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + J^{\parallel}_{\mathcal{W}_i/c_i} \dot{\Omega}_{\mathrm{W}_i/\mathrm{B}} = u_i \tag{73}$$

$$\underbrace{\begin{bmatrix} \vec{J}_{\mathcal{C}/z} & \begin{bmatrix} J^{\parallel}_{\mathcal{W}_1/c_1} \hat{a}_1 & \cdots & J^{\parallel}_{\mathcal{W}_r/c_r} \hat{a}_r \end{bmatrix} \\ \begin{bmatrix} J^{\parallel}_{\mathcal{W}_1/c_1} \hat{a}_1^{\mathrm{T}} \\ \vdots \\ J^{\parallel}_{\mathcal{W}_r/c_r} \hat{a}_r^{\mathrm{T}} \end{bmatrix} & \begin{bmatrix} J^{\parallel}_{\mathcal{W}_1/c_1} & & \\ & \ddots & \\ & & J^{\parallel}_{\mathcal{W}_r/c_r} \end{bmatrix} \end{bmatrix}}_{=\vec{Y}} = \begin{bmatrix} \vec{L}_z - \vec{\omega}_{\mathrm{B/A}} \times \vec{H}_{\mathcal{C}/z/\mathrm{A}} \\ u \end{bmatrix} \tag{74}$$

The final step is to resolve the above in the $\mathrm{F_B}$ frame. For compactness, the big matrix on the left hand side is abbreviated as $\vec{Y}$. Let $\vec{Z} = \vec{Y}^{-1}$. We will also occasionally break up $\vec{Z}$ as

$$\vec{Z} = \begin{bmatrix} \vec{Z}_{11} & \vec{Z}_{12} \\ \vec{Z}_{21} & \vec{Z}_{22} \end{bmatrix} . \tag{75}$$

When resolved, $\vec{Z}_{11}$ is a 3×3 matrix, and $\vec{Z}_{22}$ is an $r \times r$ matrix. Generally, the moment of inertia of the wheels $J^{\parallel}_{\mathcal{W}_i/c_i}$ is much smaller than the smallest eigenvalue of the moment of inertia of the rest of the spacecraft $\vec{J}_{\mathcal{C}/z}$.

Thus, $\vec{Y}$ is approximately block-diagonal. It follows that $\vec{Z}_{11} \approx \vec{J}^{-1}_{\mathcal{C}/z}$, $\vec{Z}_{21}^{\mathrm{T}} = \vec{Z}_{12} \approx \vec{J}^{-1}_{\mathcal{C}/z} \begin{bmatrix} \hat{a}_1 & \cdots & \hat{a}_r \end{bmatrix}$,

and $\vec{Z}_{22} \approx \begin{bmatrix} \left(J^{\parallel}_{\mathcal{W}_1/c_1}\right)^{-1} & & \\ & \ddots & \\ & & \left(J^{\parallel}_{\mathcal{W}_r/c_r}\right)^{-1} \end{bmatrix}$. Thus, it is common to see applications where $\vec{Y}$ is never introduced as a single matrix and instead these approximations are used. The term $\vec{\omega}_{\mathrm{B/A}} \times \vec{H}_{\mathcal{C}/z/\mathrm{A}}$ is also often omitted if the maximum expected angular velocity is small.

4 Using Boresight Constraints

The principal constraint used in the author's other works is the function

$$h = \hat{r}_{s/o} \cdot \hat{r}_{p/o} \tag{76}$$

where o is some instrument center point (because of the scale of the problem, we can often assume o is anywhere on the spacecraft, often the center of mass), s is some point to avoid pointing the instrument towards (e.g. the Sun) and is potentially time-varying, and p is some point along the instrument boresight vector fixed in the body frame. Assume we know $\hat{r}_{s/o}\big|_{\mathrm{A}}$ and $\hat{r}_{p/o}\big|_{\mathrm{B}}$ where $\mathrm{F_B}$ is an instrument-fixed frame and $\mathrm{F_A}$ is some other frame. h is a scalar, but we need to resolve the above vectors in the same frame (any frame) to compute it. One obvious choice is the inertial frame

$$h = \hat{r}_{s/o}\big|_{\mathrm{A}}^{\mathrm{T}}\, \hat{r}_{p/o}\big|_{\mathrm{A}} = \hat{r}_{s/o}\big|_{\mathrm{A}}^{\mathrm{T}}\, \mathcal{O}_{\mathrm{A/B}}\, \hat{r}_{p/o}\big|_{\mathrm{B}} = \hat{r}_{s/o}\big|_{\mathrm{A}}^{\mathrm{T}}\, \mathcal{R}_{\mathrm{B/A}}\, \hat{r}_{p/o}\big|_{\mathrm{B}} = \hat{r}_{s/o}\big|_{\mathrm{A}}^{\mathrm{T}}\, \mathcal{R}(\widetilde{q}_{\mathrm{B/A}})\, \hat{r}_{p/o}\big|_{\mathrm{B}} \tag{77}$$

Alternatively, we could use the body frame

$$h = \hat{r}_{s/o}\big|_{\mathrm{A}}^{\mathrm{T}}\, \hat{r}_{p/o}\big|_{\mathrm{A}} = \left(\mathcal{O}_{\mathrm{B/A}}\, \hat{r}_{s/o}\big|_{\mathrm{A}}\right)^{\mathrm{T}} \hat{r}_{p/o}\big|_{\mathrm{B}} = \hat{r}_{s/o}\big|_{\mathrm{A}}^{\mathrm{T}}\, \mathcal{O}^{\mathrm{T}}_{\mathrm{B/A}}\, \hat{r}_{p/o}\big|_{\mathrm{B}} = \hat{r}_{s/o}\big|_{\mathrm{A}}^{\mathrm{T}}\, \mathcal{R}_{\mathrm{B/A}}\, \hat{r}_{p/o}\big|_{\mathrm{B}} = \hat{r}_{s/o}\big|_{\mathrm{A}}^{\mathrm{T}}\, \mathcal{R}(\widetilde{q}_{\mathrm{B/A}})\, \hat{r}_{p/o}\big|_{\mathrm{B}} \tag{78}$$

As expected, resolving in either frame yields the same result.

Because the above constraint considers only a single vector in the body frame rather than a set of three orthogonal vectors (i.e. a complete frame), much of the math for computing the constraint constants was done purely for a vector $\hat{r} = \hat{r}_{p/o}$. Since $\hat{r}$ is fixed in the body frame, the dynamics of this vector are

$$\overset{\mathrm{A}\bullet}{\hat{r}} = \cancel{\overset{\mathrm{B}\bullet}{\hat{r}}} + \vec{\omega}_{\mathrm{B/A}} \times \hat{r} = \vec{\omega}_{\mathrm{B/A}} \times \hat{r} \tag{79}$$

$$\overset{\mathrm{A}\bullet\bullet}{\hat{r}} = \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \hat{r} + \vec{\omega}_{\mathrm{B/A}} \times \left(\vec{\omega}_{\mathrm{B/A}} \times \hat{r}\right) \tag{80}$$

Instead of calculating over all $\widetilde{q}$, we run the individual constraint calculations over all $\hat{r}$, thereby removing the free variable describing the rotation about $\hat{r}$. The computations for $\vec{\omega}_{\mathrm{B/A}}$ and $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}$ are still the same as in the previous section. As with this entire document, one must be careful which frames all the above vectors are resolved in. At present, the current working version of the code resolves everything in the $\mathrm{F_B}$ frame.

5 Example Control Law

Suppose there is a body fixed vector $\hat{r}_{p/o}$ which we want to align with the inertially fixed (or time-varying and independent of $\widetilde{q}_{\mathrm{B/A}}$) vector $\hat{r}_{t/o}$. A popular control law for spacecraft attitude is a PD control law, which we can develop using an "error" quaternion.

In this method, we propose a command frame $\mathrm{F_C}$, which is the frame that results from the shortest path rotation such that $\hat{r}_{p/o}$ aligns with $\hat{r}_{t/o}$ for some point t. The shortest path is given by an axis-angle formulation. The axis is easily determined as

$$\vec{n}_t = \hat{r}_{p/o} \times \hat{r}_{t/o} \tag{81}$$

and the angle is

$$\theta_t = \text{acos}\left(\hat{r}_{p/o} \cdot \hat{r}_{t/o}\right) . \tag{82}$$

Then the quaternion from F_B to F_C, or the error quaternion is then

$$\widetilde{q}_{\text{C/B}} = \cos\left(\frac{\theta_t}{2}\right) + \hat{n}_t \sin\left(\frac{\theta_t}{2}\right) . \tag{83}$$

Alternatively, if a 3-axis attitude is specified, such as a quaternion $\widetilde{q}_{\text{C/A}}$, then one can derive the error quaternion as

$$\widetilde{q}_{\text{C/B}} = \widetilde{q}_{\text{C/A}} \otimes \widetilde{q}^*_{\text{B/A}} . \tag{84}$$

A PD control law may then be

$$\vec{L} = k_p \vec{q}_{\text{C/B}} + k_d \vec{\omega}_{\text{C/B}} = k_p \vec{q}_{\text{C/B}} + k_d (\vec{\omega}_{\text{C/A}} - \vec{\omega}_{\text{B/A}}) . \tag{85}$$

As this is a linear control law, it is common to saturate θ to a small number to retain approximate linearity. To implement this control law, we then need to resolve the above vectors in the body frame, which results in

$$\vec{L}\Big|_\text{B} = k_p \sin\left(\frac{\theta_t}{2}\right) \hat{n}_t|_\text{B} + k_d \left(\mathcal{O}_{\text{B/A}}\ \vec{\omega}_{\text{C/A}}\Big|_\text{A} - \vec{\omega}_{\text{B/A}}\Big|_\text{B}\right) . \tag{86}$$

Thus, we want to compute $\hat{n}_t$ in frame F_B, which means transforming the target vector $\hat{r}_{t/o}$ coordinates into frame F_B. Throughout this controller, it is important to keep track of which frames all the relevant variables are resolved in.

Finally, suppose we define the command frame via the shortest-path rotation, and now we want to know what the command frame quaternion is for other purposes. This is one area where the frame definition can get very important and potentially confusing, as illustrated below. We already have $\overline{q}_{\text{B/A}} = \widetilde{q}_{\text{B/A}}\big|_\text{B}$ and we likely already computed $\widetilde{w}_{\text{C/B}}\big|_\text{B}$ for use in the above controller. Then

$$\widetilde{q}_{\text{C/A}}\big|_\text{C} = \left(\widetilde{q}_{\text{C/B}} \otimes \widetilde{q}_{\text{B/A}}\right)\big|_\text{C} \tag{87}$$

$$= \widetilde{q}_{\text{C/B}}\big|_\text{C} \otimes \widetilde{q}_{\text{B/A}}\big|_\text{C} \tag{88}$$

$$= \widetilde{q}_{\text{C/B}}\big|_\text{C} \otimes \left(\widetilde{q}^*_{\text{B/C}}\Big|_\text{B} \otimes \widetilde{q}_{\text{B/A}}\big|_\text{B} \otimes \widetilde{q}_{\text{B/C}}\big|_\text{B}\right) \tag{89}$$

$$= \widetilde{q}_{\text{B/A}}\big|_\text{B} \otimes \widetilde{q}_{\text{B/C}}\big|_\text{B} \tag{90}$$

That is, the "slash-and-split" rule does not apply, as previously discussed. In my experience, I have frequently come across code where the above formula is implemented (usually without code comments), so it is good to get an intuition for when this occurs. On the other hand, intuition without understanding why the quaternions appear in this order can easily lead to disaster.

References

[1] D. S. Bernstein, A. Goel, and A. Ansari, "Geometry, kinematics, statics, and dynamics," December 2019, unpublished.

[2] B. Graf, "Quaternions and dynamics," 2008. [Online]. Available: https://arxiv.org/pdf/0811.2889.pdf

[3] M. Boyle, "The integration of angular velocity," *Advances in Applied Clifford Algebras*, vol. 27, no. 3, p. 2345–2374, May 2017.

[4] F. L. Markley and J. L. Crassidis, *Fundamentals of Spacecraft Attitude Determination and Control.* Springer, 2014.

[5] M. Karpenko, S. Bhatt, N. Bedrossian, and I. M. Ross, "Flight implementation of shortest-time maneuvers for imaging satellites," *Journal of Guidance, Control, and Dynamics*, vol. 37, no. 4, pp. 1069–1079, 2014.